nificant. The values for platinum, given for comparison, were obtained along with the silicates.

Albite and microcline have a smaller proportion of oxygen atoms than silica, and this should bring the theoretical value A_v nearer 5.96 for them than for silica. Moreover, the values of A_p given contain the thermal effect of expansion. Allowances made for these two facts bring these substances into a class with the forms of silica, and thus tend to give assurance that the silica results are not exceptional, although the data are too uncertain to add anything to the exactness of our knowledge of the variations from the theoretical curve.

Anorthite is given as a type of substances showing above 1000° a much larger increase. This increase, however, is very likely in part the effect of the latent heat from a slight amount of melting, due to a very small amount of impurity. Such effects have regularly been found in other kinds of work on silicates. But while such melting can affect the atomic heat as determined for anorthite and most other substances, which crystallize readily on cooling, it can scarcely do the same for the other silicon compounds given here, all of which are very sluggish in their changes of state, so that any slight portion of them melted at the higher temperatures would in all probability merely cool to a glass during the heat determination in the calorimeter, giving out no latent heat. Indeed, since the effect of impurities in causing melting increases rapidly with the temperature, these substances are very exceptional in their value for demonstrations at high temperatures such as that which is the subject of this paper.

Summary.—The specific heats of three forms of silica and two silicates (alkaline feldspars) determined for temperatures up to 1300°, indicate that the atomic heats at constant volume for these substances increase above theoretical value, asymptotic to 5.96, as the heats of metals have been known to do, and hence that such increase is a very general phenomenon, as has been suspected.

[1] A fuller account will be published elsewhere.

[2] Nernst, W., *Ann. Physik, Leipzig*, **36**, 1911, (430).

ON CERTAIN PROJECTIVE GENERALIZATIONS OF METRIC THEOREMS, AND THE CURVES OF DARBOUX AND SEGRE

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About a year ago appeared in these Proceedings an abstract[1] of some investigations which I was then engaged in preparing for publication. As the work progressed it expanded, and I wish to devote this note to a description

of some of my new results. The details of this discussion will appear with some of those of the former note, in extended form, in a single memoir on the general theory of surfaces.

Let P be a regular point of a non-developable surface S, and l a line in the tangent plane to S at P, but not passing through P. In my preceding note, I defined a line l' which is in the relation R to the line l with respect to the asymptotic net on S. I shall henceforth say that the lines l and l' are reciprocal to each other; they are, in fact, reciprocal polars of the osculating quadric[2] of the surface at P. The line l' passes through the point P, and does not lie in the tangent plane to S at P. If at each point of S we construct a line l' protruding from the surface, we obtain a congruence Γ'; I shall say that the congruence Γ' and the congruence Γ, formed of the lines l which are reciprocal to the lines l', are reciprocal to each other.

A projective generalization of the definition of geodesics on a surface S may easily be formulated, if the congruence of normals of the surface be replaced by a congruence Γ' composed of lines l' protruding from the surface and defined projectively in terms of the surface. There exists, in fact, a two-parameter family of curves on the surface, whose osculating planes contain the corresponding lines of the congruence Γ'. I shall, with Miss Pauline Sperry,[3] call these *union curves* of the congruence Γ'. In my previous note, I suggested that in a projective generalization of metric theorems it is desirable to preserve the important property possessed by the congruence of normals: that its developables cut the surface in a conjugate net. I also characterized a congruence Γ' having this property,[4] and uniquely determined by the surface. We may call it the *pseudo-normal congruence* of the surface, and the lines of which it is composed the *pseudo-normals* of the surface. I have recently succeeded in showing that this congruence is the most suitable substitute for the congruence of normals, in attempting to generalize the surfaces of Voss. A surface of Voss is characterized by the property, that among its geodesics exist two one-parameter families which form a conjugate net; in addition to this defining property, however, is the important one, that this conjugate net of geodesics has equal tangential invariants. I have proved that this property is merely a consequence of the fact that the developables of the congruence of normals cut the surface in a conjugate net; that is, *if there exist a conjugate net formed of union curves of a congruence Γ', and if this conjugate net have equal tangential invariants, then the developables of Γ' must cut the surface in a conjugate net.* This theorem turns out to be a consequence of the general theory of reciprocal congruences.

The definition of union curves of a congruence Γ' may be dualized so as to yield a new two-parameter family of curves, related to a congruence Γ, which we shall call *adjoint union curves*. These curves are defined as follows.[3] Let C be any curve on the surface; then the tangents to the surface which are conjugate to the tangents of C form a developable, and the edge of regression of this developable we may call the first Laplace transform of C. In fact, each

point of this edge of regression I shall call the first Laplace transform of the corresponding point of C. Let Γ be a congruence composed of lines l, one in each tangent plane of the surface. Then the curve C is said to be an *adjoint union curve* of the congruence Γ if the first Laplace transform of every point P of C lies on the corresponding line l of Γ.

A study of the relations between the union curves of a congruence Γ' and the adjoint union curves of the reciprocal congruence Γ leads to many interesting results in the general theory of surfaces and rectilinear congruences. In particular, light is thrown on a number of theorems which first appeared in my second memoir on conjugate nets.[5] In the terminology of that paper, combined with the concepts of the present note, *if* Γ' *be the axis congruence of a conjugate net, then the reciprocal congruence* Γ *is the ray congruence of the associate conjugate net.* That is, the first-mentioned conjugate net is formed of union curves of the congruence Γ', and its associate conjugate net consists of adjoint union curves of the congruence Γ which is reciprocal to Γ'. This theorem is fundamental in the study of surfaces of Voss and their projective generalization.

In general, the union curves of a congruence Γ' do not coincide with the adjoint union curves of the reciprocal congruence Γ; in fact, this can happen only on a quadric, and always does on such a surface. But on any non-ruled surface S, there are particular curves, which are union curves of certain congruences Γ', and at the same time adjoint union curves of the reciprocal congruences Γ. These curves are very important in several connections, and the property just described serves as a simple geometric characterization of them. They were first defined, in a different way, by Segre,[6] and I shall therefore call them the *curves of Segre;* they are, however, merely curves which are conjugate to certain similar curves which were previously introduced by Darboux[7] in a memoir on the contact of curves and surfaces.

The curves of Darboux were called by him *lines of quadric osculation.* He defined them as follows. Each member of the three-parameter family of quadrics which have contact of the second order with the non-ruled surface S at a regular point P cuts the surface in a curve which has a triple point at P. The three tangents at such a triple point may coincide, but, if they do, it must be in one of three directions, which Darboux calls *tangents of quadric osculation.* Associated with each of these three tangents of quadric osculation is a one-parameter family of quadrics having second-order contact with the surface at P, and cutting the surface in a curve having a triple point with coincident tangents in that particular direction of quadric osculation. There are, therefore, three one-parameter families of curves, the *curves of Darboux,* whose tangents are tangents of quadric osculation. The curve of Segre may likewise be assembled into three one-parameter families, each family being conjugate to a family of curves of Darboux.

Both Segre's and Darboux's treatments depend essentially on the notion of order of contact. It has always seemed to me to be desirable, whenever pos-

sible, to supply equivalent treatments, in such cases, which are independent of this notion; not necessarily to replace the older theorems, but to afford new theorems which are frequently of more value in practice. Moreover, it is generally difficult to define order of contact in a purely geometric way, especially projectively. Consequently the following theorem is not without interest: *on any non-ruled surface S there exist three one-parameter families of curves, the members of each family of which are union curves of a congruence* Γ' *and at the same time adjoint union curves of the reciprocal congruence* Γ. These are the curves of Segre,[6] and the curves of Darboux may be characterized as composing the three families conjugate to them.

The generalization of surfaces of Voss described above is only one of the many instances in which an important metric property of a configuration is really a particular case of a far more general projective one. I have found a number of others, connected not only with the subject of geodesics but also with apparently unrelated concepts; there appears throughout a unifying feature, however, in the notion of reciprocal congruences.

[1] Green, G. M., these PROCEEDINGS, **3**, 1917, (587–592).

[2] Wilczynski, E. J., *Trans. Amer. Math. Soc., New York*, **9**, 1908, (79–120). Cf. top of p. 83.

[3] Sperry, P., *Amer. J. Math., Baltimore*, **40**, 1918, (213–224).

[4] Green, G. M., *loc. cit.*, pp. 590–591. The congruence referred to is that generated by the lines $y\zeta$.

[5] Green, G. M., *Amer. J. Math., Baltimore*, **38**, 1916, (287–324).

[6] Segre, C., *Ac. dei Lincei, Rend., Rome*, (Ser. 5), $\mathbf{17}_2$, 1908, (409–411).

[7] Darboux, G., *Bull. Sci. Math., Paris*, (Ser. 2), **4**, 1880, p. 356.

THE RECTANGULAR INTERFEROMETER WITH ACHROMATIC DISPLACEMENT FRINGES IN CONNECTION WITH THE HORIZONTAL PENDULUM

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1. Introduction.—In the *Reports of the Carnegie Institute of Washington*, 1915, No. 229, Chap. I, part 2, pp. 30 et seq., I adduced a method for the application of the displacement interferometer to the horizontal pendulum with a graphic exhibit of the results obtained during a series of months. The concave mirror design by which the spectrum interference ellipses were made available showed a very satisfactory performance. The attainable accuracy was such that for moderate constants in the installation of the pendulum, an inclination of 3×10^{-4} seconds of arc should have been registered per vanishing interference fringe (ellipse), or about 10^{-3} seconds per 10^{-4} cm. of displace-